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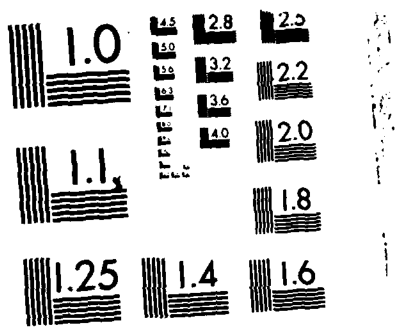

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NOTE ON THE DISPLACEMENT EFFECT IN A SPINNING FLUID

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ABSTRACT

The interpretation of viscous effects in perturbations of rapid, solid-body rotation of a fluid in a cylindrical gyroscope is discussed. This is bound up with the explanation of the roots of those effects and aims to improve the foundation on which more sophisticated questions in this field can be addressed successfully. To help with the clarification of viscous displacement effects, their explanation in Section 3 is prefaced by a discussion of two examples displaying some of the salient features in a simpler setting.

AMS (MOS) Subject Classifications: 76-02, 76U05, 76D10

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SIGNIFICANCE AND EXPLANATION

A number of technical devices, such as some governors, navigational instruments and projectiles, for example, employ liquid-filled gyroscopes. It has long been known that the motion of the liquid can upset the operation of such a device by means of the forces and moments exerted by the liquid on its solid container. That fact has prompted studies of the possible perturbations of the motion which the fluid is meant to perform in unison with its container, first of all under conditions in which the contributions to the forces and moments arising from the viscosity of the fluid can be expected to be small in relative magnitude.

A gyroscope, however, is an intricate dynamical mechanism that can react in rather startling ways to quite small moments, if those are out of step with the strong, stabilizing influence of the spin. Analysis of the viscous effects shows that they can act out of step with the main fluid motions and therefore require more attention than their magnitude would appear to justify.

The following notes respond to a request for a clarification of some rather complicated, conceptual issues that arise in the analysis of 'small' viscous effects even in the case where the container is a cylinder in steady rotation and the fluid moves almost in unison with it. The request reflects a realization that a clearer understanding of the issues in this simplest case is needed as a basis for fruitful thought about the still trickier issues arising for different modes of operation and for different container shapes of practical importance for such devices.

To divide the difficulties, the response is made in three stages, of which the last addresses the viscous effects localized near the end-walls of a spinning cylinder. The first two Sections serve to introduce some of the salient issues in the much simpler setting of two examples involving oscillation, but no rotation.

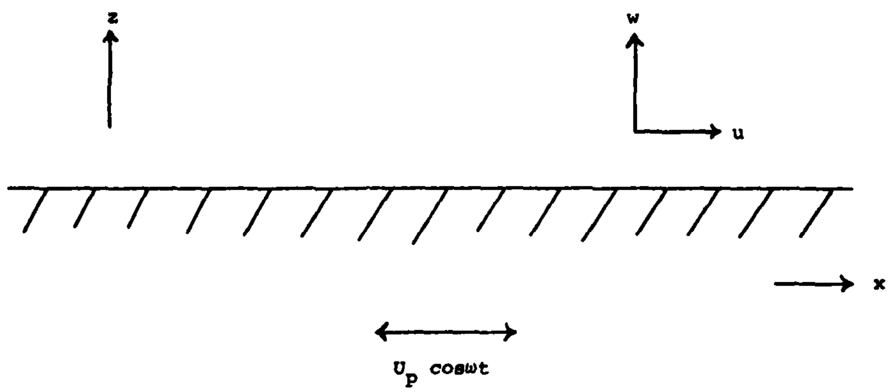
The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

NOTE ON THE DISPLACEMENT EFFECT IN A SPINNING FLUID

R. E. Meyer

1. First Example

Envisage an unbounded fluid resting on a horizontal plate $z = 0$ which oscillates in its own plane. The fluid is incompressible and Newtonian, so viscous shear will transmit motion from the plate to the fluid, but that fluid motion can be expected to decay with increasing distance z from the plate.



Since no origin of the horizontal coordinates is distinguishable in the plane of the plate, the fluid velocity cannot depend on the coordinates x,y. The plate is envisaged to be in rectilinear oscillation, so that every point of it has velocity

$$U_p \cos\omega t = \mathrm{Re}\{U_p e^{i\omega t}\}$$

in the x-direction, say, with real constant velocity U_p and real constant frequency ω. Nobody blows from far away, so by symmetry, the fluid can have no horizontal velocity component perpendicular to the x,z-plane of Fig. 1. The mass-conservation equation,

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$\operatorname{div} \underset{\sim}{v} = 0$, reduces exactly to $\partial w/\partial z = 0$ because nothing depends on x and y. But on the plate (envisaged impermeable), $w = 0$, so

$$w \equiv 0 .$$

In the incompressible Navier-Stokes equations, most terms are now seen to vanish identically, leaving only

$$\partial u/\partial t = \nu \partial^2 u/\partial z^2 , \tag{1}$$

where ν is the kinematic viscosity. The boundary conditions are

$$\begin{aligned} u &= U_p e^{i\omega t} \qquad && \text{at } z = 0 , \\ u &\to 0 && \text{at } z \to \infty . \end{aligned} \tag{2}$$

Initial conditions are not here considered, because the simplest question of interest concerns the motion of permanent character that could develop ultimately. The familiar development variable $z^2/(4\nu t)$ is not then relevant; instead, the boundary conditions (2) show that permanent character is possible only with a motion periodic in time, and the linearity of (1), (2) precludes any nontrivial solution of frequency other than ω. Hence, $u/e^{i\omega t}$ can depend only on z. But, the problem has no reference length other than $(\nu/\omega)^{1/2}$, so the motion can depend only on the nondimensional distance

$$z\,(\omega/2\nu)^{1/2} = \eta \tag{3}$$

and the velocity must have the form

$$u(z,t) = U_p f(\eta) e^{i\omega t} . \tag{4}$$

Substitution of (4) into (1) yields

$$2if = f''$$

and the boundary conditions (2) show that

$$f = e^{-(1+i)\eta} , \tag{5}$$

$$\begin{aligned} u &= U_p \operatorname{Re}\{e^{-\eta + i(\omega t - \eta)}\} \\ &= U_p e^{-\eta} \cos(\omega t - \eta) . \end{aligned} \tag{6}$$

This exact solution of the Navier-Stokes equations is due to Stokes (1851). It illustrates two very typical features of oscillatory or rotating boundary layers. The first is the appearance of a complex exponent in (5). Its real part is not unexpected as

a description of the amplitude decay of the fluid oscillation with increasing distance from the plate. The imaginary part is seen from (6) to represent a phase lag of the fluid oscillation, relative to the plate oscillation, which increases in proportion to the distance from the plate. Such phase delay in the fluid oscillation turns out to be a very typical manifestation of the viscous diffusion of vorticity in oscillating or rotating boundary layers.

The second feature typical of boundary layers caused primarily by oscillation or rotation is the length scale

$$(2\nu/\omega)^{1/2} = \delta$$

characteristic of boundary layer structure in the direction normal to the wall. It is the basic boundary layer thickness because any specific choice of a definition of such thickness must clearly yield just a numerical multiple of δ.

On the other hand, Stokes' very simple solution is atypical in that here the normal velocity w vanishes identically throughout the fluid. Accordingly, no definite displacement thickness can be defined, nor any rational "boundary layer thickness" other than δ.

2. Second Example

For an illustration of further, typical features in the simplest setting, envisage now the same geometry, but with the plate fixed (and still impermeable), while a distant contraption imparts a sloshing motion to the fluid. It is natural to expect such a motion to be non-uniform in space, but if the contraption is long in one direction, the motion should retain essentially the symmetry which makes the velocity independent of the y-coordinate and makes the y-component of velocity vanish everywhere. For present purposes, the sloshing motion can then be specified adequately by saying that, if the fluid were inviscid, its x-velocity at the plate would be

$$u_e(x,t) = U_s\, u_0(x/d)\, \cos\omega t \qquad (7)$$

with constant, real frequency ω, velocity-amplitude scale U_s, and length scale d related to a dimension of the distant contraption or the large tank holding the fluid.

For a brief remark, suppose the fluid were inviscid, then the mass-conservation equation

$$\partial u/\partial x + \partial w/\partial z = 0 \qquad (8)$$

implies already that $w \not\equiv 0$, and by (7), there must be a vertical velocity

$$w_e = -(U_s/d)\, u_0'(x/d)\, (z + c)\, \cos\omega t \qquad (9)$$

close to the plate. The integration 'constant' $c(x,t)$ is zero if the boundary condition $w = 0$ at the plate be imposed on this inviscid motion, but our interest in a real fluid suggests a postponement of this step until more is learned about the structure of the transition to the no-slip boundary condition.

Even for an inviscid fluid, however, the specification (7) already implies definition of a non-dimensional velocity ratio

$$U_s/(\omega d) = Ro\ ,$$

called Rossby number in Geophysics. In the present context, it is natural to think of d as a 'large' length scale, and if the frequency ω is 'not small', then the Rossby number should have a small value.

For a Newtonian fluid, an additional, nondimensional Ekman number

$$\nu/(\omega d^2) = E$$

appears, which will take even much smaller values when the viscosity is 'small', so that the Reynolds number

$$Re = U_s d/\nu = Ro/E$$

is then large.

A rational way to formulate the second example nondimensionally is to measure horizontal velocity u in units of U_s, time in units of ω^{-1}, and horizontal distance x in units of d, as in (7), but vertical distance z again in units of $\delta = (2\nu/\omega)^{1/2}$ to clarify the transition from the sloshing motion to no-slip on the plate. Mass-conservation (8) then implies that vertical velocity w must be measured in units of $U_s\delta/d$, as usually in boundary layers. The proper unit of pressure, by contrast, is not so straightforward. The first example confirmed the plausible expectation that oscillation, per se, does not generate spatial pressure gradients, and if they occur, they must therefore relate to those of the sloshing, which can be estimated in a limit $\eta \to \infty$, $z \to 0$. That limit can be taken so as to be governed plausibly by inviscid equations, of which the x-momentum conservation law is

$$\partial u_e/\partial t + u_e \partial u_e/\partial x + w_e \partial u_e/\partial z = -\rho^{-1}\partial p_e/\partial x \ ,$$

in the dimensional notation of (7) and (9). From those two equations, $u_e \to U_s u_0 \exp(i\omega t)$ and $w_e \to -U_s c(\partial u_0/\partial x)\exp(i\omega t)$ as $\eta \to \infty$, but $z \to 0$, and accordingly,

$$-\frac{\partial p_e}{\partial x} = \rho\omega U_s e^{i\omega t}\left\{iu_0 + Ro\,\frac{\partial u_0}{\partial(x/d)}\left[u_0 - \frac{c}{d}\,\frac{\partial u_0}{\partial(z/d)}\right]e^{i\omega t}\right\} \ . \tag{10}$$

[For just a passing moment, u_0 is here admitted to depend also on z.] For consistency with the scaling of vertical velocity w just deduced from mass conservation, moreover, the constant c, if nonzero, must scale with δ. Since present interest centers on cases where Ro and δ/d are not at all large, the proper unit of pressure p is seen to be $\rho\omega d U_s$.

If all the variables are now made nondimensional by reference to the units just specified, the mass-conservation equation (8) remains unchanged and the momentum-conservation equations of a Newtonian fluid take the form

$$\frac{\partial u}{\partial t} + Ro\left(u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial \eta}\right) = -\frac{\partial p}{\partial x} + E\frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\frac{\partial^2 u}{\partial \eta^2}\,, \tag{11}$$

$$\frac{\partial w}{\partial t} + Ro\left(u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial \eta}\right) = -\frac{1}{2E}\frac{\partial p}{\partial \eta} + E\frac{\partial^2 w}{\partial x^2} + \frac{1}{2}\frac{\partial^2 w}{\partial \eta^2}\,. \tag{12}$$

The parameters Ro and E are seen to play quite different roles. For 'small viscosity' or high frequency, $E \to 0$ and then (11) assumes boundary layer character and (12) shows vertical pressure variation across the boundary layer to be negligible, as usually, so that $p \sim p(x,t)$, which is predictable from the limit $\eta \to \infty$, $z \to 0$, i.e., from the non-dimensional form of (10),

$$-\frac{\partial p}{\partial x} = \frac{\partial}{\partial t}\left(\frac{u_e}{U_s}\right) + Ro\; u_0'(x)[u_0 + O(E^{1/2})]e^{2it} \tag{13}$$

where u_e and u_0 match the notation in (7), (9).

Comparison of (13) and (10) clarifies why the verical variation of the sloshing velocity plays no role in the small-Ekman limit: that variation is on the length scale d and the analysis of the boundary layer is concerned with variations on the length scale $\delta = (2E)^{1/2}d$, on which the sloshing appears independent of x and z. This explains why the specification (7) of the limit $z \to 0$ of the sloshing velocity turns out sufficient for the analysis of the boundary-layer transition. Furthermore, the x-dependence of the transition becomes a parametric effect: the transition depends only on the local values of u_e and $\partial u_e/\partial x$.

For small Rossby number $Ro = U_s/(\omega d)$, moreover, a first approximation to the boundary layer structure is now seen to be obtainable from a linear truncation of (11),

$$\partial u/\partial t - U_s^{-1}\partial u_e/\partial t = \frac{1}{2}\,\partial^2 u/\partial\eta^2 \ , \tag{14}$$

almost as simple as Stokes' equation (1) for the first example. The boundary conditions for it appropriate to the description of a transition from the sloshing (7) as $\eta \to \infty$, but $z \to 0$, to no-slip at the plate $\eta = 0$ are

$$\begin{aligned} &u = 0 \quad \text{for} \quad \eta = 0 \\ &u \to u_e/U_s = u_0(x)e^{it} \quad \text{as} \quad \eta \to \infty \ , \end{aligned} \tag{15}$$

for all x,t.

The linearity of (14), (15) again rules out any frequency other than ω and in fact, makes u/u_e independent of x and t, so that

$$U_s u(x,\eta,t)/u_e(x,t) = e^{-it}u(x,\eta,t)/u_0(x) = g$$

can depend only on η and must satisfy

$$g'' = 2i(g - 1), \quad g(0) = 0, \quad g(\eta) \to 1 \quad \text{as} \quad \eta \to \infty \ ,$$

whence $g = 1 - e^{-(1+i)\eta}$ and

$$u = u_0(x)e^{it}(1 - e^{-(1+i)\eta}) \ . \tag{16}$$

Thus, $[1 - g(\eta)]u_e = u_e - U_s u$ exhibits the same diffusive decay and phase lag as $f(\eta)$ in (5).

The object of the second example is the interpretation of the vertical velocity w, which is now determined by (8) and the boundary condition $w = 0$ at the impermeable plate $\eta = 0$, so that

$$w(x,\eta,t) = -\int_0^\eta \frac{\partial u}{\partial x}\,d\eta' = -u_0'\,e^{it}\int_0^\eta g(\eta')d\eta'$$

$$= -u_0'(x)\;e^{it}[\eta - \frac{1}{1+i} + 2^{-1/2}e^{-\eta-i(\eta+\pi/4)}] \ . \tag{17}$$

An initially disturbing feature here is the first term, which grows beyond bounds as $\eta \to \infty$, but actually, the first two righthand terms in (17) can be recognized as precisely the nondimensional form of (9). The last term describes an internal boundary layer effect, which decays and lags diffusively with increasing distance from the wall, and w is seen to trail u by another $\pi/4$ in phase.

The most interesting term in the bracket of (17) is the second, because it describes a boundary layer effect which persists as $\eta \to \infty$ and even, for $z > 0$ outside the boundary layer. It describes a contribution to the vertical velocity which is independent of distance from the plate, but oscillary in time, and not in phase with the sloshing velocity u_e. In the efficient, complex notation of (17), it is accordingly represented by a complex multiple $u_0'(x)/(1 + i)$ of $\exp(it)$, of which the η-independence connotes persistence beyond the boundary layer, the modulus describes amplitude, and the argument specifies phase difference.

It should be observed in retrospect that everything that has [illegible] said here about the vertical velocity sprang directly from the mass-conservation equ[illegible]n (8). The proper, physical interpretation of the effect under discussion is theref[illegible] in terms of a mass-flow contribution to the sloshing motion arising from the viscou[illegible]sion of vorticity close to the plate. This is the interpretation which goes to the [illegible]art of the matter.

On the other hand, if one desires to be done with boundary layer details and to look only at their resultant effect in terms of a corrected 'inviscid' description of the sloshing motion, then the last term in (17) becomes immaterial and that equation reduces to

$$w = -(U_s/d)u_0'(x/d)e^{i\omega t}[z - \delta/(1 + i)] \tag{18}$$

in dimensional, but still complex, notation. As already noted, that is the complex version of (9), and there are two alternative, simple interpretations. A kinematical one is that the plate $z = 0$ appears porous to the 'inviscid eye' with an oscillatory suction velocity. That suction velocity is proportional to the limit of $\partial w_e/\partial z$ as $z \to 0$ and it therefore shares the spatial structure of w_e. Its temporal structure is different, however, because the factor of $-\partial w_e/\partial z$ is $2^{-1/2}\delta \exp(-i\pi/4)$; it lags behind $-w_e$ by $\pi/(4\omega)$.

A more common, kinematic-geometrical interpretation is in terms of the displacement thickness δ_1 introduced by Prandtl, probably in response to early audiences sceptical of his novel and controversial notions and particularly, in response to their demand for a

concrete explanation of his order-of-magnitude concept of boundary-layer thickness. The displacement thickness is defined as the distance $z = \delta_1$ from the geometrical plate position at which the corrected 'inviscid' description (18) of the Newtonian fluid motion satisfies the classical wall-condition $w = 0$. Hence,

$$\delta_1 = \delta/(1 + i) = (2\nu/\omega)^{1/2}/(1 + i) = (\nu/\omega)^{1/2}e^{-i\pi/4} \tag{19}$$

by (18), and this displacement thickness is complex because it characterizes a viscous mass-flow contribution that is out of phase with the main sloshing motion.

It will be clear enough that this is just a simple example of a mass-flow effect characteristic of oscillating or rotating boundary layers. Of course, it is a small contribution when $\delta/d = (2E)^{1/2}$ is a very small number, but 'small' must depend on the practical question at issue. Some issues, such as stability limits, are notoriously sensitive to phase relations, and the 'small' viscous mass-flow contribution is then important because it is the one and only part of the Newtonian motion outside the boundary layer that is out of phase with the imposed sloshing (7).

It may be remarked finally that the real part of (16) and (17) is

$$u = u_e(x,t) - U_s u_0(x/d)e^{-z/\delta}\cos(\omega t - z/\delta) ,$$

$$w = -\frac{U_s}{d} u_0'\left(\frac{x}{d}\right)\left[z \cos \omega t + \left(\frac{\nu}{\omega}\right)^{1/2}\cos\left(\omega t + \frac{3\pi}{4}\right) + \left(\frac{\nu}{\omega}\right)^{1/2} e^{-z/\delta}\cos\left(\omega t - \frac{z}{\delta} - \frac{\pi}{4}\right)\right] ,$$

$$\delta = (2\nu/\omega)^{1/2} ,$$

and illustrates why calculations prefer to aim at its simpler, complex version (19). In the real version, the definition of δ_1 makes $\delta_1 = -c$, which fluctuates between positive and negative values.

That c turns out independent of x is seen in retrospect to have two reasons: First, the analysis of the transition to no-slip concerns only the very local dynamics, in the limit $E \to 0$, and is influenced only parametrically by the x-dependence of the sloshing, so that $u_e(x,0,t)$ and $-\partial u_e/\partial x$ at $(x,0,t)$ become the respective scale-constants of the horizontal velocity and vertical velocity gradient for the local

dynamics. Secondly, the linearity of the limit $Ro \to 0$ makes the velocity components directly proportional to their scales; by suppressing nonlinear interactions, it eliminates the more complicated structure of the harmonics of the sloshing frequency ω from consideration.

It may also be worth repetition that the two small parameters Ro and E are seen to play quite different roles in the analysis. A double-limit $Ro \to 0$, $E \to 0$ arises, and no consideration has been given to the manner in which an experiment might approach it. Instead, the choice has been made to let $Ro \to 0$ first and then look at (11), (12) for $Ro = 0$ and $E \ll 1$. The same, somewhat arbitrary choice is repeated in what follows.

3. Ekman Layer in a Cylinder

A question of practical importance concerns the perturbations of solid-body rotation of a fluid in a spinning and nutating cylinder, because of the oscillating forces and moments which the fluid may exert on its container. The following focuses on the boundary layers on the end-walls of the cylinder, because the obvious fact that they make a very small contribution to the perturbations at small Ekman numbers can be misleading: As in the preceding examples, viscous diffusion of momentum is associated with a delay, in which case the viscous contribution to the perturbations may be out of phase with the 'main' fluid oscillations and a comparison of magnitudes may be pointless in regard to questions for which phase relationships are decisive. The objective of the dicussion will be to clarify why this raises issues which are more complicated than, but in their decisive features, closely analogous to, those just discussed for the second example.

Envisage, then, a cylinder of radius a spinning with angular velocity Ω about the z-axis and filled with fluid above the plane end-wall $z = 0$. For the study of the stability of solid rotation, it is convenient to refer the motion to cylindrical polar coordinates r,θ,z in the frame of an observer rotating with the cylinder, who sees only the perturbation from the solid rotation, but also, a Coriolis acceleration. He can absorb the centrifugal force by counting as his pressure p the difference between the real pressure and the centrifugal contribution $\frac{1}{2}\rho\Omega^2 r^2$ + const to it.

It is basic for stability analysis to consider velocity perturbations of scale U_s small compared with the velocity Ωa of the cylinder (and such that the space gradient of velocity has only the scale U_s/a and the time-rate of change of velocity, only the scale ΩU_s). In short, there is a Rossby number $Ro = U_s/(a\Omega) \ll 1$ and to that extent, the motion is "geostrophic" and a straightforward application of the limit $Ro \to 0$ will again linearize the equations of motion and will thereby ignore interactions between harmonics. That offers an opportunity for a relatively simple, modal analysis by separation of all the variables so that the resultant velocity field is the real part of

$$\begin{bmatrix} u_e \\ v_e \\ w_e \end{bmatrix} = \begin{bmatrix} \hat{u}(r)\cos kz \\ \hat{v}(r)\cos kz \\ \hat{w}(r)\sin kz \end{bmatrix} e^{i(Ct-m\theta)} \tag{20}$$

where r, z are made nondimensional by reference to a and t, by reference to Ω^{-1}, and the velocity components are measured in units of U_s. [It is assumed, for simplicity, that the height and diameter of the cylinder do not differ so much that their ratio introduces an additional small parameter.] The dependence on z is left real to display that the modal analysis is inviscid to the extent of recognizing the impermeability of the wall $z = 0$, but ignoring the no-slip condition there.

In (20), m is integer and C is the nondimensional, complex frequency of the perturbation, and it should be remarked briefly that the analysis in the frame of a laboratory - fixed observer differs only in the interpretation of that frequency. For that observer, the phase is constant when θ increases at an additional unit rate, so that his exponent in (20) is $i(C + m)t - im\theta$ in the present notation.

The Newtonian fluid, however, has also an Ekman number

$$E = \nu/(a^2\Omega) ,$$

and when that is small, all experience [Greenspan 1968] indicates that the transition to no-slip occurs again in an Ekman layer of thickness scale

$$\delta = (2\nu/\Omega)^{1/2} = a\ (2E)^{1/2} .$$

To study that transition, requires again consideration of the limit

$$z \to 0, \quad E \to 0 \quad \text{for fixed} \quad \eta = z/\delta \in (0,\infty) . \tag{21}$$

Mass-conservation shows, as in the second example, that the corresponding, vertical velocity w' must be measured in units $\delta U_s/a = U_s(2E)^{1/2}$. The issues raised by the pressure scale are entirely analogous to those arising in the second example: the same argument shows the scale of p to be $\rho\Omega a U_s$. With these scale revisions, the Navier-Stokes equations assume boundary layer character, and the vertical momentum equation shows that pressure differences in the vertical direction within the Ekman layer can be only $O(E)$. The limit (21) of the pressure perturbation is therefore that at $z = 0$ in the

inviscid modal analysis leading to (20), i.e.,

$$-\partial p/\partial r = \rho\left[\partial u_e/\partial t - 2v_e\right]_{z=0} + O(Ro) \ ,$$
$$-r^{-1}\partial p/\partial\theta = \rho\left[\partial v_e/\partial t + 2u_e\right]_{z=0} + O(Ro) \ . \tag{22}$$

With horizontal velocity components u, v still measured in units of U_s, the limit $Ro \to 0$ linearizes the conservation equations for horizontal momentum and the limit (21) then reduces them to

$$\begin{aligned}\partial u/\partial t - 2v &= -\rho^{-1}\partial p/\partial r + \frac{1}{2}\partial^2 u/\partial\eta^2 \\ \partial v/\partial t + 2u &= -\rho^{-1}r^{-1}\partial p/\partial\theta + \frac{1}{2}\partial^2 v/\partial\eta^2\end{aligned} \tag{23}$$

where the pressure gradient is given by (22) and hence, depends only on r, θ and t. The transition equations for

$$u - u_e(r,\theta,0,t) = \tilde{u} \quad \text{and} \quad v - v_e(r,\theta,0,t) = \tilde{v}$$

are therefore the typical, simple Ekman equations of geostrophic theory,

$$\begin{aligned}\partial\tilde{u}/\partial t - 2\tilde{v} &= \frac{1}{2}\partial^2\tilde{u}/\partial\eta^2 \\ \partial\tilde{v}/\partial t + 2\tilde{u} &= \frac{1}{2}\partial^2\tilde{v}/\partial\eta^2\end{aligned} \tag{24}$$

with boundary conditions

$$\begin{aligned}u \equiv u_e + \tilde{u} = v \equiv v_e + \tilde{v} = w' = 0 \quad &\text{at} \quad \eta = 0 \ , \\ \tilde{u}, \tilde{v} \quad \text{both} \ \to 0 \quad \text{as} \quad \eta \to \infty, \quad &\text{but} \quad z \to 0 \ .\end{aligned} \tag{25}$$

These show again that the Ekman transition depends on r and θ only parametrically through the effective, local scales u_e and v_e of the linear system (24), (25). However, u and v are not directly proportional to u_e and v_e, respectively, because of the Coriolis coupling in (24). Instead,

$$\partial(\tilde{u} \pm i\tilde{v})/\partial t \pm 2i(\tilde{u} \pm i\tilde{v}) = \frac{1}{2}\partial^2(\tilde{u} \pm i\tilde{v})/\partial\eta^2$$

and the boundary conditions show

$$u \pm iv = (u_e \pm iv_e)[1 - \exp(\lambda_\pm \eta)] \tag{26}$$

with

$$\lambda_+^2 = 2i(C + 2), \qquad \lambda_-^2 = 2i(C - 2)$$

and λ_+, λ_- understood as the roots of negative real part -- which are well-defined, unless both $\mathrm{Re}\ C = \mp 2$ and $\mathrm{Im}\ C > 0$, and that exceptional case will not be considered here.

In sum, the Coriolis-coupling is left intact by the geostrophic limit $Ro = 0$, and this gives a much more complicated structure to the transition from no-slip to the horizontal velocity (u_e, v_e) of the modal analysis at $z = 0$. Each of u and v has two contributions to the transition, one of local scale $U_s u_e$, the other, of local scale $U_s v_e$. Each contribution, moreover, displays a double structure, one of boundary layer thickness δ/λ_+ and the other, of thickness δ/λ_-. The difference in the thicknesses, moreover, is accompanied by a difference in diffusive phase lag. All the same, the transition shares with that of the second example the feature that it is complete for u and v, i.e., no contribution of order $E^{1/2}$ to either is left as $\eta \to \infty$. A mass-flow effect of that order, however, appears as possible here as in the second example, and should show up in the normal velocity w.

Substitution of (26) into the mass-conservation equation

$$\frac{\partial}{\partial r}(ru) + \frac{\partial v}{\partial \theta} + r\frac{\partial w}{\partial z} = 0 \tag{27}$$

gives

$$\frac{\partial w}{\partial z} = [1 + \frac{1}{2}(e^{\lambda_+ \eta} + e^{\lambda_- \eta})]\frac{\partial w_e}{\partial z} + \frac{1}{2i}(e^{\lambda_+ \eta} - e^{\lambda_- \eta})\zeta_e \tag{28}$$

in the limit (21), where ζ_e stands for

$$\zeta_e = \frac{1}{r}\frac{\partial}{\partial r}(rv_e) - \frac{\partial u_e}{r\partial \theta}, \tag{29}$$

which is the z-component of (perturbation) vorticity. The Coriolis-coupling is seen from (28) to generate also two distinct contributions to w, one of the scale $U_s w_e$, just as in the second example, but the other, of scale a times the normal component of the perturbation vorticity. The generation of such vorticity by the Coriolis effect is, of course, a well-known geophysical phenomenon.

Since $\partial w_e/\partial z$ and ζ_e take their values at $z = 0$ in the limit (21), the first approximation to the value of w as $\eta \to \infty$, but $z << 1$ is found from (28) by integration to be

$$w = [\, z - \frac{1}{2}\,\delta(\lambda_+^{-1} + \lambda_-^{-1})](\partial w_e/\partial z)|_{z=0} + \frac{1}{2}\, i\delta(\lambda_+^{-1} - \lambda_-^{-1})\zeta_e|_{z=0} \tag{30}$$

and this represents the part of the normal velocity which persists beyond the boundary layer.

Now, the analysis of the Ekman boundary layer is tailored to the premise that the subscript e distinguishes quantities varying on the scales Ω^{-1}, a and U_s characteristic of time, distance and velocity, respectively, of the core-flow outside the boundary layers on the solid walls and outside that region around the rim of the end-walls in which those boundary layers meet. Variation on those scales implies that the (first approximation to the) equations governing the core-flow are the limit $Ro = E = 0$ of (11) and (12). In the cylindrical coordinates of the rotating observer, those governing equations are (27) and

$$\partial u_e/\partial t - 2v_e = -\partial p_e/\partial r \ ,$$
$$\partial v_e/\partial t + 2u_e = -r^{-1}\partial p_e/\partial \theta \ ,$$
$$\partial w_e/\partial t = -\partial p_e/\partial z$$

and the velocities (20) in the core-flow are effectively computed from them. Cross-differentiation to eliminate the pressure from the first two of that equation-trio yields

$$\partial(r\zeta_e)/\partial t = 2r\partial w_e/\partial z \ ,$$

by (29) and (27), and since the time-factor of ζ_e is also $\exp(iCt)$, by (29) and (20),

$$i\zeta_e = (2/C)\partial w_e/\partial z$$

and (30) simplifies to

$$w = [z - \frac{1}{2}\,\delta(\lambda_+^{-1} + \lambda_-^{-1}) + \delta C^{-1}(\lambda_+^{-1} - \lambda_-^{-1})]\ \partial w_e/\partial z|_{z=0} \ . \tag{31}$$

This has again the two interpretations noted in the second example. The purely kinematical one is that the main effect of the end-wall transition on the core-flow is to make it look as if there were a small, very judicious amount of blowing and sucking,

$$w - w_e = -\delta(\partial w_e/\partial z)|_{z=0}[\frac{1}{2}\,(\lambda_+^{-1} + \lambda_-^{-1}) + C^{-1}(\lambda_-^{-1} - \lambda_+^{-1})] \ ,$$

through the wall $z = 0$, which shares the spatial structure of w_e, but differs in phase by, essentially, the argument of the complex-valued square bracket. The more geometrical interpretation of (31) is of a wall displaced to the value $z = \delta_1$ at which the normal velocity (31) vanishes, which is

$$\delta_1 = (\frac{1}{2}\nu/\Omega)^{1/2}[(\lambda_+^{-1} + \lambda_-^{-1}) + (2/C)(\lambda_-^{-1} - \lambda_+^{-1})] ,$$

as first determined by Kitchens, Gerber and Sedney [1978]. This coordinate $z = \delta_1$ is complex because it is the geometric interpretation of a mass-flow from the end-wall boundary layer that is out of phase with the core-flow structure of a purely inviscid fluid. That phase-difference, in turn is the reason why the mass-flow contribution from the viscous vorticity-diffusion influences eigenvalues to an extent out of proportion to its small magnitude.

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The interpretation of viscous effects in perturbations of rapid, solid-body rotation of a fluid in a cylindrical gyroscope is discussed. This is bound up with the explanation of the roots of those effects and aims to improve the foundation on which more sophisticated questions in this field can be addressed successfully. To help with the clarification of viscous displacement effects, their explanation in Section 3 is prefaced by a discussion of two examples displaying some of the salient features in a simpler setting.

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